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Blunt-body impact on a compressible liquid surface

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In this paper we are concerned with the unsteady plane liquid motion due to the penetration of a blunt undeformable contour through the free surface. Initially the liquid is at rest, and the contour touches its free surface at a single point. At the initial stage of the process the liquid motion is described within the framework of the acoustic approximation. It is known that, just behind the shock front which is generated under the impact, the liquid motion does not depend on the presence of the free surface for all time. The pressure distribution and the velocities of liquid particles inside this region are calculated analytically for an arbitrary contour. It is shown that liquid motion close to the contact points just before the shock wave escapes onto the free surface is self-similar; the singularity of the pressure is analysed. The focusing of the shock wave generated by the impact of a body with a shallow depression in the front surface is discussed.

1. Introduction

In this paper we consider the problem of the plane unsteady flow arising when a blunt contour enters an ideal weakly compressible liquid through the free surface. Until some time, which is taken as the initial one ($t = 0$), the liquid occupies a lower half-plane and is at rest. The line $y = 0$ corresponds to the undisturbed position of the free liquid surface. Initially the body touches this line at a single point, which is taken as the origin of the Cartesian coordinate system. Then the body starts to penetrate the liquid vertically with a constant velocity V (figure 1). The topology of the liquid boundary changes at the initial moment: a previously absent component of the liquid boundary adjacent to the solid body appears. The presence of the contact points between the free surface and the solid one is the main characteristic of the liquid–solid impact problem. The positions of these point must be determined together with the solution of the problem.

We shall determine the liquid flow, the elevation of the free surface, the position of the contact points and the pressure distribution under the following assumptions: (i) the body is solid, undeformable, smooth, and symmetrical with respect to the y-axis; (ii) the radius of curvature at the contour top differs from zero; (iii) the fluid is ideal and compressible, and its motion is plane and symmetrical with respect to the y-axis; (iv) the Mach number $M = V/c_0$, where c_0 is the sound velocity in the fluid at rest, is much less than unity; (v) external mass forces and surface tension are absent. The symmetrical case is considered for simplicity of notation only. All results, except some special cases, are valid for an arbitrary contour.

A detailed review of the subject has been given by Lesser & Field (1983), who consider the problem of a drop impact onto a solid surface. The water-entry problem is included in Korobkin & Pukhnachov's (1988) review but the main part of that

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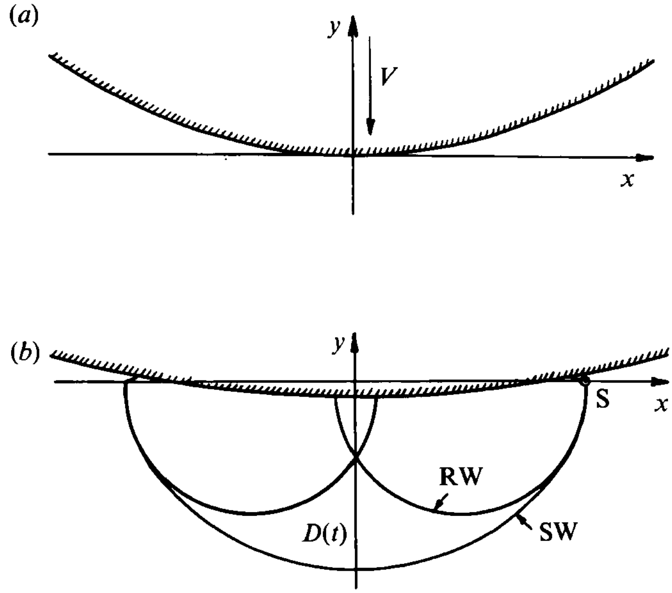

FIGURE 1. Impact by a blunt contour on a liquid surface. (*a*) Initially, liquid is at rest and a body touches its free surface at the single point. (*b*) The flow pattern at the subsonic stage: SW, shock wave; RW, wave produced by the shock reflection at the free surface; $D(t)$, region under consideration in the present paper.

review is devoted to the incompressible liquid model. However, it can be shown that for the initial stage of the solid–liquid impact, when the size of the contact region is small in comparison with the characteristic linear size of the process geometry, the problem can be approximately transformed to the water-entry problem, which has a simpler geometry than the original one.

Let the initial position of the free surface close to the first contact point be given by the equation $y = h_i(x)$, and the position of the solid surface for all time by the equation $y = f(x) - Vt$. The functions $h_i(x), f(x)$ are assumed to be smooth and $f(0) = h(0) = 0, f'(0) = h'(0) = 0$. Then the transformation $y_1 = y - h(x)$, $x_1 = x$ maps the original position of the free surface onto the horizontal line $y_1 = 0$, but the position of the solid surface is now described by the equation $y_1 = [f(x_1) - h(x_1)] - Vt$. It can be verified that this mapping does not change the equations of motion, boundary and initial conditions to leading order with respect to the contact spot size. This is why the present results are valid not only for the problem under consideration but also for the more general liquid–solid impact problems.

It is well known (see Bowden & Field 1964) that within the framework of the above-mentioned assumptions there is an instant T such that for $0 < t < T$ the free surface is undisturbed. This stage of the impact process may be called supersonic. Its presence is connected with the fact that the expansion velocity of the contact spot for small times is greater than the local sound velocity. This result follows from purely geometrical considerations and it fails within the framework of a more complete model which takes into account viscosity of the liquid. Nevertheless, for the model of a weakly viscous, compressible fluid a similar initial stage, during which the disturbances of the free surface are localized close to the solid contour, can also be indicated. For large Reynolds numbers the size of this region is inversely proportional to *Re*.

When the Mach number is small, an approximate model describing the ideal liquid flow caused by the entry of a parabolic contour at the supersonic stage and just after it has been constructed by Korobkin (1990) using Lagrangian approach. This approach to the water-entry problem was first used by Pukhnachov (1979). The generalization of the model to an arbitrary contour, with $T > 0$ as the only limitation, is not difficult. It is clear, that at the initial stage, when $t/T = O(1)$, the same approximate model can be used for an arbitrary blunt contour. Namely, one needs to find the velocity potential $\phi(x, y, t)$ that satisfies the wave equation in the lower half-plane ($y < 0$), and the mixed boundary conditions on the line $y = 0$, and which is identically equal to zero when $t < 0$. The positions of the points which divide the boundary between the contact spot and the free surface (contact points) are known at the supersonic stage and are determined by a transcendental equation when $t > T$ (Korobkin 1990).

For all times there is a region $D(t)$ in the half-plane $y < 0$ (figure 1*b*) where the liquid motion does not depend on the presence of the free surface and is the same as in the problem of a body emerging from an infinite plate. The geometry of this region, the pressure distribution and the velocity field inside it are under investigation. For the supersonic stage this analysis has been given by Rochester (1979) for the similar problem of cylindrical drop impact onto a rigid plane. Unfortunately his results are yet unpublished. The present paper may be considered as the generalization of Rochester's results to an arbitrary geometry of the blunt contour and to all times. This generalization is not trivial and allows the possibility of analysing some very interesting effects caused by the special geometry of the entering body. The shape of the body can be chosen in such a way that the focusing of the shock wave under a solid–liquid impact will be observed. The phenomenon can be utilized for kidney stone disintegration, and the present approach is expected to be very helpful for its optimization.

The main characteristics of the evolution of the shock wave under solid–liquid impact and pressure distribution along the shock front can be described within the framework of geometrical acoustic theory (see Lesser 1981). However, to construct a general picture of the process it is desirable to have more detailed information about the liquid motion under the impact than we can obtain using this theory. This is why the acoustic theory, which is more general than the geometrical acoustic one is used in the present paper. The acoustic theory allows us to calculate analytically the velocities of liquid particles and the pressure field everywhere and to analyse their characteristics.

2. Formulation of the problem

Non-dimensional variables are used below. They are chosen so that after scaling both the sound velocity in the liquid at rest and the impact velocity are equal to unity in the new variables. The notation of §1 is unchanged. The boundary-value problem for the velocity potential $\phi(x, y, t)$ written in Eulerian variables is

$$\phi_{tt} = \phi_{xx} + \phi_{yy} \quad (y < 0),$$
$$\phi = 0 \quad (y = 0, |x| > a(t)),$$
$$\phi_y = -1 \quad (y = 0, |x| < a(t)),$$
$$\phi = \phi_t = 0 \quad (y < 0, t = 0),$$
$$\phi \to 0 \quad (y^2 + x^2 \to \infty).$$

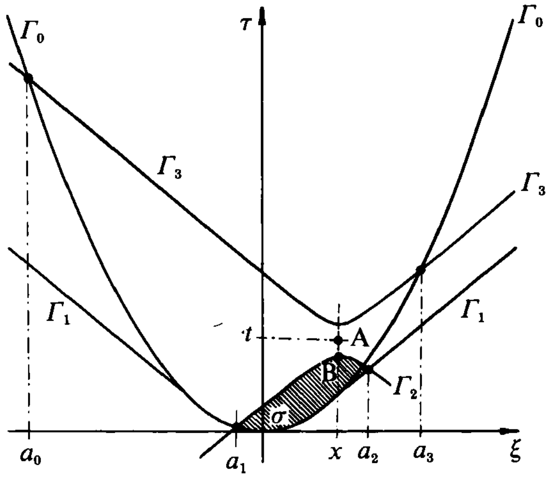

FIGURE 2. Geometry of the integration domain.

Here the interval $-a(t) < x < a(t)$ corresponds to the wetted part of the entering contour. The function $a(t)$ is assumed to be known. A method for its calculation has been proposed by Korobkin (1990). In our case $a(0) = 0$, $a'(t) > 1$ at the supersonic stage $(0 < t < T)$, $a'(t) < 1$ at the subsonic stage $(t > T)$, and $a'(T) = 1$ at the moment of the shock escape onto the free surface. The contour position is given in non-dimensional variables by the equation $y = \alpha(M)\,(f(x)-t)$, where the function $\alpha(M)$ tends to zero when $M \to 0$. At the supersonic stage $f[a(t)] = t$. For example, for a parabolic contour $f(x) = \frac{1}{2}x^2$ and it can easily be found that $a(t) = (2t)^{\frac{1}{2}}$, $T = \frac{1}{2}$.

The velocity potential is calculated in accordance with the well-known formula

$$\phi(x,y,t) = \frac{1}{\pi}\iint_{\sigma(x,y,t)} \frac{\phi_y(\xi,0,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau}{[(t-\tau)^2-(x-\xi)^2-y^2]^{\frac{1}{2}}}, \tag{1}$$

where the integration domain $\sigma(x,y,t)$ on the plane ξ,τ is bounded below by the curve

$$\Gamma_1 : \tau = f(\xi),\ |\xi| < a_*,$$
$$\tau = |\xi| + T - a_*,\ |\xi| > a_*,$$

$a_* = a(T)$, and above by the curve

$$\Gamma_2 : \tau = t - [(x-\xi)^2 + y^2]^{\frac{1}{2}} =: F(\xi, x, y, t). \tag{2}$$

The coordinate of the left-hand intersection point of the curves Γ_1 and Γ_2 is denoted by a_1 and that of the right-hand point by a_2 (see figure 2). It is clear that a_1, a_2 are functions of the variables x, y, t. In figure 2 the curve Γ_3 is the mirror image of Γ_2 with respect to the line $\tau = t$ and the curve Γ_0 is defined by the equation $\tau = f(\xi)$. For the curves Γ_0 and Γ_3 the ξ-coordinate of the left-hand intersection point is denoted by a_0, and that of the right-hand intersection point by a_3.

In (1) the function $\phi_y(\xi, 0, \tau)$ in the general case is unknown when $\tau > T$. However for the interval $0 < \tau < T$

$$\phi_y(\xi,0,\tau) = -1,\ |\xi| < a(\tau), \quad \phi_y(\xi,0,\tau) = 0,\ |\xi| > a(\tau).$$

Therefore the velocity potential and its first derivatives are given by quadratures in some region of the motion domain, where the inequalities $-a_* < a_1 < a_2 < a_*$ are satisfied. It will be shown that for every pair (x, t) an interval of the vertical variable y can be found, where the above-mentioned inequalities are valid.

Let some point A on the plane ξ, τ with coordinates $x, t (x > 0)$ (see figure 2) be fixed. Let the point B lie at the top of the hyperbola Γ_2 so that $AB = -y$. In general the slope of the curve Γ_2 is

$$\frac{\partial F}{\partial \xi} = \frac{x-\xi}{[(x-\xi)^2+y^2]^{\frac{1}{2}}}.$$

It is obvious, that this value is less than or equal to unity and that equality occurs when $y = 0$. We then begin to increase y, starting from zero. The hyperbola Γ_2 will be lowered and flattened. For some value $y = Y_1(x, t)$ we will have $a_1 > -a_*, a_2 = a_*$. When $y < Y_1(x, t)$ both intersection points lie below the line $\tau = T$ and, hence, for such values of the depth the fluid motion does not depend on the presence of the free surface. On further increasing the depth y the intersection points of the curves Γ_1, Γ_2 continue to move together and for some $y = Y(x, t)$ they coalesce. The value of ξ at which this happens will be denoted by $s(x, t)$. Therefore, for $t > 0$ a region $D(t)$ of the (x, y)-plane can be distinguished, where the presence of the free surface is not important. When $y < Y(x, t)$ the liquid is at rest, and so one can say that the equation $y = Y(x, t)$ determines the disturbance front or the shock wave front. Behind this front, where $y = Y(x, t)+0$, the pressure is, in the general case, not zero.

3. The geometry of the region $D(t)$

The function $Y_1(x, t)$ for $x > 0$ is determinated by the condition $a_2 = a_*$ which, with the help of (2), leads to the equation

$$(x-a_*)^2+Y_1^2(x, t) = (t-T)^2. \tag{3}$$

When $t < T$ it is necessary to take $Y_1 = 0$. Hence, the upper boundary of the region $D(t)$ consists of the circular arc with radius $(t-T)_+$ and centre at the point $(a_*, 0)$, and the part of the solid boundary $0 < x < a(t)$ when $0 < t < T$ and $0 < x < (a*+T-t)_+$ when $t > T$. The notation $g_+ = g$ when $g > 0$ and $g_+ = 0$ when $g < 0$ is used.

When $\xi = s(x, t)$ the determination of the function $Y(x, t)$ yields the following equations to be satisfied:

$$\left.\begin{aligned} f(s) &= F(s, x, Y(x, t), t), \\ \frac{\mathrm{d}f}{\mathrm{d}\xi}(s) &= \frac{\mathrm{d}F}{\mathrm{d}\xi}(s, x, Y(x, t), t).\end{aligned}\right\} \tag{4}$$

The first one describes the intersection of the curves Γ_1 and Γ_2, the second one their smooth touching at this point. The equations determine the position of the shock front in the parametrical form

$$y = -(t-f(s))\,[1-[f'(s)]^2]^{\frac{1}{2}}, \tag{5a}$$

$$x = s+f'(s)\,(t-f(s)), \tag{5b}$$

where s now plays the role of the parameter. It is clear that the same result could be obtained using Huygen's principle as was done by Lesser (1981). Most important here are the inequalities for the parameter $-a(t) < s < a(t)$ when $0 < t < T$ and $-a* < s < a_*$ when $t > T$. They show that the form of the shock front for all time depends only on the contour geometry in the small vicinity of the apex.

The boundaries of the region $D(t)$ are determined by (3) and (5). For the symmetrical case, the vertical width of the region is greatest when $x = 0$, and during the supersonic stage it increases with time from zero to T. For $T < t < T + a_*$ it continues to increase and achieves the maximum value $a_* + T$ when $t = a_* + T$. Then it decreases and tends to T when $t \to \infty$.

The shock front position close to the right-hand contact point just before the end of the supersonic stage is now analysed. It is convenient to use 'internal' variables λ, μ, τ such that

$$x = a(t) + \lambda, \quad t = T + \tau, \quad y = \mu,$$

where

$$\lambda < 0, \quad \mu < 0, \quad \tau < 0, \quad \lambda^2 + \mu^2 + \tau^2 \ll 1.$$

Note that the dimensions of this area are unknown and have to be determined together with the construction of the asymptotic formula. For this we will use the function $Y(x, t)$ in the parametrical form (5), where we must put $s = a(t) + \kappa$, where κ is a new parameter such that $|\kappa| \ll 1$, $\kappa < 0$. Taking into account the equality $f''(a_*) = -a''(T-0)$ and the notation $f_2 = f''(a_*)$ we get

$$t - f(s) = T + \tau - f(a(T+\tau) + \kappa) = -\kappa[1 + f_2(\tau + \tfrac{1}{2}\kappa)] + O(|\tau|^3 + |\kappa|\tau^2),$$
$$f'(s) = 1 + f_2(\tau + \kappa) + O(\tau^2 + |\kappa\tau|).$$

It is clear that in the general case one has to take $\kappa = O(\tau)$. Then to leading order when $|\tau| \to 0$ the system (5) gives

$$\left.\begin{aligned} Y &= -(-\tau)^{\frac{3}{2}} k[2f_2(1+k)]^{\frac{1}{2}} + \ldots, \\ \lambda &= -\tau^2 f_2\, k(2 + \tfrac{3}{2}k) + \ldots; \end{aligned}\right\} \tag{6}$$

$k = \kappa/\tau$ is the new parameter, $k > 0$. Equations (6) show that the shock front evolution has a self-similar character in the region under consideration. Namely, there is some function $L(x)$ such that the $Y(x,t)$ can be written in the form

$$Y = (-\tau)^{\frac{3}{2}} L(\lambda/\tau^2)\,(1 + o(1))$$

when $\tau \to 0$, $\lambda = O(\tau^2)$.

Let us consider the geometry of the region $D(t)$ close to the point S where it is joined to the free surface (see figure 1), that is, where

$$x = a_* - T + t + \lambda, \quad y = \mu, \quad \lambda < 0, \quad \mu < 0, \quad \lambda^2 + \mu^2 \ll 1.$$

We will seek the positions of its upper boundary (expansion wave front) and its lower boundary (shock wave front) in the forms $\lambda = \lambda_e(\mu, t)$ and $\lambda = \lambda_s(\mu, t)$ respectively. Then (3) is reduced to

$$\lambda_e^2 + 2(t-T)\,\lambda_e + \mu^2 = 0,$$

and, when $\mu \to 0$,

$$-\lambda_e = \frac{\mu^2}{2(t-T)} + \frac{\mu^4}{8(t-T)^3} + O(\mu^6). \tag{7}$$

To obtain the related expression for the shock front, we use the parameter κ such that $s = a_* + \kappa$, where $\kappa < 0, |\kappa| \ll 1$. Equation (5*a*) leads to the following asymptotic expansion:

$$\kappa(\mu, t) = A_1(t)\,\mu^2 + A_2(t)\,\mu^4 + O(\mu^6);$$

its substitution into (5*b*) gives the final formula

$$-\lambda_s = \frac{\mu^2}{2(t-T)} + \frac{\mu^4}{8(t-T)^3}\left(1 - \frac{1}{f_2(t-T)}\right) + O(\mu^6), \tag{8}$$

Thus only the fourth derivatives of these curves differ. The higher order of the touching of the expansion and shock fronts indicates that, inside this region, the pressure gradient can be unbounded and, hence, the assumptions of the acoustic approximation fail. It may be expected that the fine structure of the liquid flow in this region can be described using the 'short wave' approximation proposed by Ryzhov & Christianovich (1958) for compressible flows with small, but abrupt, variations of the pressure. In any case, at some distance, which is large with respect to the size of the vicinity of the point S, the geometry of the region $D(t)$ is determined by the relations (7) and (8), and the flow is close to that calculated within the framework of the acoustic approximation.

4. The pressure distribution inside the region $D(t)$

If the point with coordinates x, y at the moment t lies inside $D(t)$ then (1) can be written as

$$\phi(x,y,t) = -\frac{1}{\pi}\int_{a_1}^{a_2}\left(\int_{f(\xi)}^{F(\xi)} \frac{\mathrm{d}\tau}{[(t-\tau)^2-(x-\xi)^2-y^2]^{\frac{1}{2}}}\right)\mathrm{d}\xi.$$

The inner integral is calculated analytically. It is equal to zero when $\xi = a_1$ or a_2 since $f(\xi) = F(\xi)$ for those values (figure 2). This is why under the differentiation of (1) with respect to the variables x, y, t the differentiation and the integration with respect to ξ are interchangeable. The final formula is

$$p(x,y,t) = \frac{1}{\pi}\int_{a_1}^{a_2} \frac{\mathrm{d}\xi}{[(t-f(\xi))^2-(x-\xi)^2-y^2]^{\frac{1}{2}}}, \tag{9}$$

where a_1, a_2 are functions which are determined geometrically and depend on x, y, t. The integrand is an elementary acoustic source which is located at the point on the liquid boundary lying a distance ξ from the point of first contact and which acts at the instant $f(\xi)$. This is the instant when the liquid particle touches the solid contour for the first time. One may say that the pressure field in the region $D(t)$ is formed by the elementary 'explosions' of the liquid particles at the instant of their first contact with the solid body. The same arguments have been used by Lesser (1981) for the construction of the pressure distribution along a shock front from a physical point of view within the framework of geometrical acoustics. The present analysis generalizes Lesser's result to the whole region behind the shock front.

For a parabolic contour $f(\xi) = \frac{1}{2}\xi^2$ the expression under the radical sign in (9) is a polynomial of the fourth degree with respect to ξ with coefficients depending on x, y, t. We denote this polynomial by $P_4(\xi)$. Clearly $P_4(a_j) = 0$, $j = 0, 1, 2, 3$ and $P_4(\xi) > 0$ when $a_1 < \xi < a_2$. The formula (9) now takes the form

$$p(x,y,t) = \frac{2}{\pi}\int_{a_1}^{a_2} \frac{\mathrm{d}\xi}{[(a_3-\xi)(a_2-\xi)(\xi-a_1)(\xi-a_0)]^{\frac{1}{2}}}.$$

This integral is a special one and we obtain

$$p(x,y,t) = (4/\pi)\,\zeta(x,y,t)\,K(k(x,y,t)), \tag{10}$$

where $K(k)$ is a complete elliptic integral of the first kind,

$$\zeta(x,y,t) = ((a_3-a_1)(a_2-a_0))^{-\frac{1}{2}},$$
$$k(x,y,t) = [(a_2-a_1)(a_3-a_0)]^{\frac{1}{2}}\zeta(x,y,t).$$

Thus, to find the pressure field in the region $D(t)$ for the parabolic contour case we have to find $a_j(x,y,t)$ $(j=0,1,2,3)$ – this is a geometrical problem – and then to calculate the value of the function $K(k)$. When $0<t<T$ and $y=0$, (10) coincides with that obtained by Rochester (1979) for the pressure distribution over the wetted part of a plane during the supersonic stage of a drop impact onto this plane. When $t>0$ and $y\to Y(x,t)+0$ we have

$$a_2-a_1\to 0, \quad k\to 0, \quad K(k)\to \tfrac{1}{2}\pi,$$

and hence

$$p(x,Y(x,t)+0,t)=2[(a_3-s)(s-a_0)]^{-\frac{1}{2}},$$

where the function $s(x,t)$ was determined above. This formula describes the pressure distribution along the back of the shock front for an arbitrary time.

Let us construct a similar formula for an arbitrary case. We put $y=Y(x,t)+\Delta$ in (9) and consider the limit of the right-hand side when $\Delta\to+0$. A new function $g(\xi,x,y,t)$ is introduced by the relation

$$g(\xi,x,y,t)(\xi-a_1)(a_2-\xi)=(t-f(\xi))^2-(x-\xi)^2-y^2. \tag{11}$$

The function is positive in the integration interval $a_1<\xi<a_2$ and

$$g(a_j,x,y,t)=2[x-a_j-f'(a_j)(t-f(a_j))]/(a_2-a_1), \tag{12}$$

where $j=1,2$. For some bounded function $B(x,t)$, the exact form of which is not important, the following asymptotic expansions are valid when $\Delta\ll 1$:

$$a_1=s-B(x,t)\Delta^{\frac{1}{2}}+O(\Delta), \quad a_2=s+B(x,t)\Delta^{\frac{1}{2}}+O(\Delta).$$

The relations can be obtained using the conditions (4) and double Talor series of the left-hand sides of the equations

$$G(a_j,t,x,\Delta)=0$$

close to the points $a_j=s(x,t)$, $\Delta=0$, where

$$G(a_j,t,x,\Delta)=(t-f(a_j))^2-(x-a_j)^2-(Y(x,t)+\Delta)^2.$$

Then

$$\frac{1}{2}\frac{\partial^2 F}{\partial a_j^2}(s,t,x,0)(a_j-s)^2+\frac{\partial G}{\partial \Delta}(s,t,x,0)\Delta=O(|a_j-s|^3+|a_j-s|\,|\Delta|)$$

and we obtain the above-mentioned asymptotic expansions. Substitution of those asymptotic expansions into (12) yields the final relation

$$\lim_{\Delta\to 0} g(a_j,x,y,t)=1-[f'(s)]^2+f''(s)(t-f(s)).$$

Taking this analysis into account we can rewrite (9) when $|\Delta|\ll 1$ as

$$p(x,y,t)=\frac{1}{\pi}\int_{a_1}^{a_2}\frac{1}{[g(\xi,x,y,t)]^{\frac{1}{2}}}\frac{\mathrm{d}\xi}{[(\xi-a_1)(a_2-\xi)]^{\frac{1}{2}}}$$

$$\approx\frac{1}{\pi}\frac{1}{[g(s(x,t),x,Y(x,t),t)]^{\frac{1}{2}}}\int_{a_1}^{a_2}\frac{\mathrm{d}\xi}{[(\xi-a_1)(a_2-\xi)]^{\frac{1}{2}}},$$

which gives

$$p_{\mathrm{sw}}(s,t)=[1-[f'(s)]^2+(t-f(s))f''(s)]^{-\frac{1}{2}}, \tag{13}$$

where $p_{\mathrm{sw}}(s(x,t),t)=p(x,Y(x,t)+0,t)$. Equation (13) describes the pressure distribution along the back of the shock wave and has to be considered together with the system (5). For example, for the parabolic contour when $f(\xi) = \frac{1}{2}\xi^2$ we have

$$p_{sw} = [1+t-\tfrac{3}{2}s^2]^{-\frac{1}{2}}, \quad y = -(t-\tfrac{1}{2}s^2)(1-s^2)^{\frac{1}{2}}, \quad x = s(1+t-\tfrac{1}{2}s^2),$$

where $|s| < (2t)^{\frac{1}{2}}$ when $0 < t < \frac{1}{2}$ and $|s| < 1$ when $t > \frac{1}{2}$. For an arbitrary symmetrical contour at the shock-front top ($x = 0$), (13) gives

$$p_{sw}(0,t) = [1+tf''(0)]^{-\frac{1}{2}}.$$

This means that the parabolic contour case is a boundary case. When $f(x) = O(x^{2+k})$, $x \to 0$, $k > 0$, i.e. when the contour is 'more blunt' than a parabola, then $f''(0) = 0$ and the pressure at the shock top is equal to the 'hydraulic hammer' pressure for all time. When $f(x) = O(x^{1+k})$, $x \to 0$, $0 < k < 1$, i.e. when the contour is 'less blunt' than a parabola, then $f''(0) = \infty$ and the pressure jump at the top of the shock wave is equal to zero.

At the contact point during the supersonic stage when $f(s) = t$, (13) gives

$$p_{sw}(a(t),t) = [1-f'(a(t))]^2]^{-\frac{1}{2}},$$

or in a more convenient form

$$p_c(s) = v_c/(v_c^2-1)^{\frac{1}{2}}, \tag{14}$$

where v_c is the velocity of the contact point, $p_c(s) = p_{sw}(s,f(s))$. Equation (14) shows that the pressure very close to the contact points is unbounded when $v_c \to 1+0$.

Let us consider the pressure field in this region in detail. It is convenient to use the 'internal' variables λ, μ, τ determinated by the equations

$$x = a(t)+\lambda, \quad y = \mu, \quad t = T+\tau.$$

When $\lambda^2+\mu^2+\tau^2 \to 0$ the three intersection points of the curves $\Gamma_0, \Gamma_2, \Gamma_3$ merge (see figure 2), that is

$$a_1 \to a_* - 0, \quad a_2 \to a_* - 0, \quad a_3 \to a_* + 0.$$

To find the asymptotics of $a_j(x,y,t)$ ($j = 1,2,3$) inside the region, the equation

$$(t-f(a_j))^2-(x-a_j)^2-y^2 = 0 \tag{15}$$

has to be rewritten using the new 'internal' variables, and the new unknown function $\sigma(\lambda,\mu,\tau) = a_j - a(t), |\sigma| \ll 1$ must be introduced. On decomposing the left-hand side of (15) in power series firstly with respect to σ and then with respect to τ, while neglecting terms of the higher order since $|\sigma| \ll 1, |\tau| \ll 1$, we find

$$\begin{aligned} t-f(a_j)+x-a_j &= t-f[a(t)+\sigma]+\lambda-\sigma \\ &= t-f[a(t)]-f'[a(t)]\sigma+\lambda-\sigma+O(\sigma^2) = -2\sigma+\lambda+O(\sigma^2+|\sigma\tau|), \\ t-f(a_j)-x+a_j &= [1-f'[a(t)]]\sigma-\tfrac{1}{2}f_2\sigma^2-\lambda+O(\sigma^3) \\ &= -f_2\sigma(\tau+\tfrac{1}{2}\sigma)-\lambda+O(\sigma^2[|\sigma|+|\tau|]). \end{aligned}$$

The resulting equation has to be at least cubic. This is why we need to assume

$$\sigma/\tau \sim 1, \quad \lambda/\tau^2 \sim 1, \quad \mu/(-\tau)^{\frac{3}{2}} \sim 1.$$

Then (15) takes the form

$$f_2 k^2(2+k)+2k\frac{\lambda}{\tau^2}+\left[\frac{\mu}{(-\tau)^{\frac{3}{2}}}\right]^2 = O(\tau),$$

where $k(\lambda,\mu,\tau)=\sigma(\lambda,\mu,\tau)/\tau$. It can be shown that inside the disturbed part of the liquid domain bounded below by the shock front this cubic equation has three real roots: $k_1>k_2>0>k_3$. Then $a_j=a(t)+k_j(\lambda/\tau^2,\mu/(-\tau)^{\frac{3}{2}})\tau+O(\tau^2), j=1,2,3$ and the asymptotics of the pressure when $\tau\to 0$ is

$$p(x,y,t)=\frac{1}{\pi}[v_c'(T)\tau]^{-\frac{1}{2}}\int_{k_2}^{k_1}\frac{\mathrm{d}k}{[(k_1-k)(k-k_2)(k-k_3)]^{\frac{1}{2}}}[1+o(1)],$$

or in more compact form

$$p(x,y,t)=\frac{2}{\pi}[v_c'(T)(k_1-k_3)\tau]^{-\frac{1}{2}}K\left[\left(\frac{k_1-k_2}{k_1-k_3}\right)^{\frac{1}{2}}\right][1+o(1)]. \tag{16}$$

The pressure depends on the contour form via the acceleration of the contact point at the escape moment, that is $v_c'(T)$. Equation (16) is valid for an arbitrary non-pointed contour with the only restriction that $v_c'(T)$ is not equal to zero.

5 The velocity field in the region $D(t)$

The velocity vector $\boldsymbol{u}=\nabla\phi$ can be found in the same manner as in the previous section:

$$\boldsymbol{u}(x,y,t)=\frac{1}{\pi}\int_{a_1}^{a_2}\frac{(x-\xi,y)}{(x-\xi)^2+y^2}\frac{(t-f(\xi))\,\mathrm{d}\xi}{[(t-f(\xi))^2-(x-\xi)^2-y^2]^{\frac{1}{2}}}. \tag{17}$$

To obtain the liquid velocity on the shock front it is necessary to consider (17) when $y\to Y(x,t)+0$, $a_1\to s(x,t)-0$, $a_2\to s(x,t)+0$. Then for all time

$$\boldsymbol{u}(x,Y(x;t)+0;t)=p_{\mathrm{sw}}(s,t)\,\boldsymbol{n}(s), \tag{18}$$

where $\boldsymbol{n}$ is the unit external normal to the shock front. As expected, the velocity has the magnitude of the pressure jump and its tangential component is continuous on the shock front.

To find the velocity field close to the contact points just before the shock escape onto the free surface we use the same 'internal' variables as in the previous section. But

$$\frac{(x-\xi)(t-f(\xi))}{(x-\xi)^2+y^2}=1+o(1)$$

when $\tau\to 0$. Therefore the main terms of the asymptotic expansions of the pressure and the horizontal velocity coincide. For the vertical velocity component it is convenient to rewrite (17) in the form

$$v=\frac{1}{\pi}\int_{a_1}^{a_2}\frac{y\,\mathrm{d}\xi}{y^2+(x-\xi)^2}+\frac{1}{\pi}\int_{a_1}^{a_2}\frac{y[t-f(\xi)+[g(\xi)(\xi-a_1)(a_2-\xi)]^{\frac{1}{2}}]^{-1}\,\mathrm{d}\xi}{[g(\xi)(\xi-a_1)(a_2-\xi)]^{\frac{1}{2}}\{t-f(\xi)+[g(\xi)(\xi-a_1)(a_2-\xi)]^{\frac{1}{2}}\}}.$$

The relation (11) shows that the expression in the curly brackets is close to $a(t)-\xi$ when $\tau\to 0$. Reasoning in the same way as when deriving (16) we obtain

$$v=-1+\frac{1}{\pi}|v_c'(T)|^{-\frac{1}{2}}\left[\frac{\mu}{(-\tau)^{\frac{3}{2}}}\right]\int_{k_2}^{k_1}\frac{\mathrm{d}k}{k[(k_1-k)(k-k_2)(k-k_3)]^{\frac{1}{2}}}+o(1),$$

where the functions $k_j(\lambda/\tau^2,\mu/(-\tau)^{\frac{3}{2}}), j=1,2,3$ have been determined above.

The final result can be presented as follows. In the region under consideration the liquid motion and the pressure distribution are approximately self-similar and are described by the formulae

$$\left.\begin{aligned} u(x,y,t) &= (-\tau)^{-\frac{1}{2}}U\left(\frac{\lambda}{\tau^2},\frac{\mu}{(-\tau)^{\frac{3}{2}}}\right)+\dots, \\ v(x,y,t) &= V\left(\frac{\lambda}{\tau^2},\frac{\mu}{(-\tau)^{\frac{3}{2}}}\right)+\dots, \\ p(x,y,t) &= u(x,y,t)+\dots, \end{aligned}\right\} \tag{19}$$

where the functions U, V are bounded and continuous in the flow domain.

6. Comparison with Lesser's results

The impact of a liquid drop onto both a rigid and an elastic plane has been considered by Lesser (1981) for both the plane and the axisymmetric case. Using geometrical acoustic theory he gives the evolution of the shock front and the pressure distribution along it for the supersonic stage. In §1 it was remarked that, for arbitrary geometry of both the body and the liquid volume at the initial contact moment with the only limitation that those surfaces are smooth, the impact problem is equivalent to the water-entry problem. It is evident that the shock front evolution in the acoustic theory and in the geometrical acoustic theory are the same, as both theories are based on Huygen's principle. We want to show that the present results concerning the pressure distribution along the shock front agree with those of Lesser.

Using the parametrical definition of the shock front position (5) it can be found that the unit external normal vector $\boldsymbol{n}$ at the front is given by

$$\boldsymbol{n} = [f'(s), -(1-[f'(s)]^2)^{\frac{1}{2}}]$$

and is independent of time. This means that if we fix a point $(s,0)$ on the interval $-a_* < x < a_*, y = 0$ and take a ray starting from this point, the direction of which is given by the vector $\boldsymbol{n}(s)$, then the ray will be perpendicular to the front at the intersection point for all time (see figure 3). The distance along the ray from the initial point to the front is equal to $t-f(s)$ at the moment t. But $f(s)$ coincides with the instant of time when the liquid particle, initially on the free surface at the distance s from the coordinate origin, reaches the solid surface of the entering contour. One may now say that along every ray, indicated by the parameter s, the shock front is displaced independently, with constant velocity which is equal to unity in the non-dimensional variables, and its motion starts at time $f(s)$. This is one of the main assumptions in geometrical acoustic theory. The result is non-trivial because the shock wave is not free but forced by the contact point motion.

Equation (18) shows that just after the shock front has passed through a liquid particle it attains a velocity directed along the ray which is equal to the pressure jump on the shock front. Taking (13), (14) into account we find

$$p_{\text{sw}}(s,t) = p_{\text{c}}(s)\left(\frac{R(s)}{R(s)+t-f(s)}\right)^{\frac{1}{2}}, \tag{20}$$

where $$R(s) = (1-[f'(s)]^2)/f''(s)$$

is the radius of curvature of the shock front at the contact point at the instant $f(s)$. In this form the formula for the pressure jump coincides with Lesser's, obtained within the framework of the geometrical acoustics for the drop impact problem.

Lesser's analysis is based on simple and clear physical reasoning, but the geometry

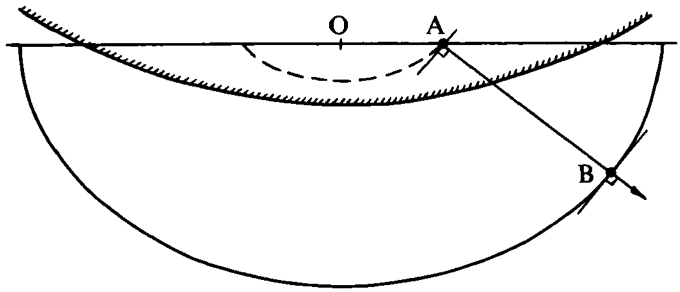

FIGURE 3. Ray picture at instant t: ——, shock wave; ----, shock wave at the instant $f(s)$; →, rays; $|OA| = s$, $|AB| = t - f(s)$.

of the impact process that he considered is not the simplest one. In particular, the expression for the radius of curvature $R(s)$ was not given and numerical calculation was suggested. In the present formula (13) the pressure jump on the shock front depends on the body geometry in an explicit way. This is why we can now consider the propagation of the shock wave initiated by the impact in detail and analyse the characteristics caused by special geometrics of the entering contour.

7. The focusing of the shock wave initiated by impact

The focusing of the shock wave under a solid–liquid impact was first observed by Dear & Field (1988). They used a gel piece with a shallow depression in the front surface and impacted it with a flat-fronted slider at 150 ms^{-1}. High-speed photography shows that there is no point focusing and that is strong.

In this section the process of the entry of a body with a depression at the top into a liquid is considered. At the initial moment the body touches the liquid surface at two points as shown in figure 4. The slope of the tangent to the contour between those points is small. In dimensional variables it must be less than the Mach number. In this case the 'internal' part of the free surface will be at rest for all time. Then the shock position and the pressure distribution along the back of the shock front are given by (5), (13) which are valid not only for convex contours, like parabolas, but also for contours having parts with negative curvature, where $R(s) < 0$ for some values of the parameter s. It was shown in the previous section that the influence of the body geometry on the shock wave propagation can be analysed independently for every ray. Therefore we can first consider some simple contour forms and then construct a contour with given properties using those elementary ones.

Inside the region $D(t)$ under consideration in this paper, $|f'(s)|$ is less than unity and $\operatorname{sgn}(R(s)) = \operatorname{sgn}(f''(s))$, and so

$$1-[f'(s)]^2 > 0, \quad t-f(s) > 0$$

in the expression under the radical in (13). Therefore for some fixed s such that $f''(s) < 0$ we can always find the instant $t_f(s)$ when this expression will be equal to zero. This is the focusing moment for this ray and we cannot continue our calculations along the ray within the framework of the linear acoustic theory.

We have two curves on the plane (s,t). The first one is given by the equation $t = f(s)$. It coincides with the form of the entering contour and indicates when a liquid particle, initially on the free surface at a distance s from the coordinate origin, meets the solid surface. The second one,

$$t = t_f(s) = f(s) - (1-[f'(s)]^2)/f''(s), \tag{21}$$

gives the focusing time for a fixed value of s.

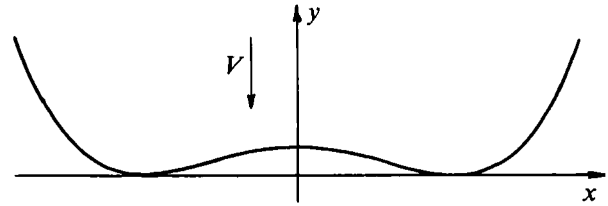

FIGURE 4. Impact by a body with a shallow depression at the top.

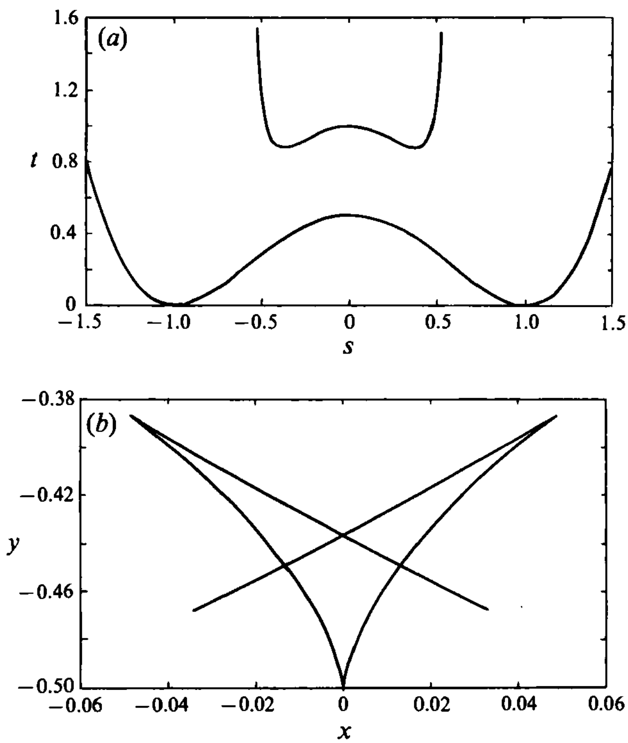

FIGURE 5. Focusing of the shock wave under the entry of a body of the geometry shown in figure 4. (*a*) Focusing time (top curve) and first contact time. (*b*) Focusing points.

These curves, for the function $f(s) = \frac{1}{2}(s^2-1)^2$, which is chosen only for simplicity of calculation, and which describes a contour with a depression at the top, can be seen in figure 5(*a*). The position of the focusing points for the given contour is shown in figure 5(*b*). In the general case this curve is given in parametric form by

$$y(s) = \frac{1}{f''(s)}(1-[f'(s)]^2)^{\frac{3}{2}}, \quad x = s-\frac{f'(s)}{f''(s)}(1-[f'(s)]^2), \tag{22}$$

which can be obtained by substitution of (21) into the system (5). For the geometry under consideration there are three special points (see figure 5*b*). These are the points where the tangent vector to the curve reverses direction. In the vicinities of these points one expects the highest values of the pressure.

The question is, can the form of the entering contour be chosen in such a way that both the focusing moments $t_f(s)$ and the focusing point positions $x(s), y(s)$ are the same for all values of s? This question can be put in a simpler form if we limit

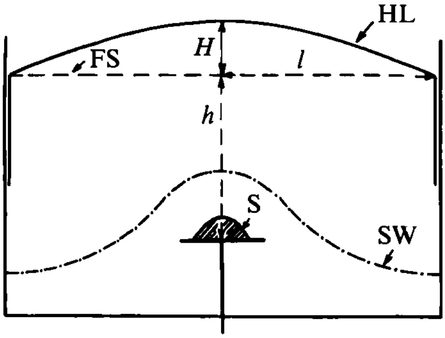

FIGURE 6. Double-focusing device. Schematic representation and notation: SW, position of the shock wave just before the focusing instant; FS, initial position of the free surface; HL, hyperbolic lid; S, stone.

ourselves to the focusing moments. Then we need to find function $f(s)$ such that the right-hand side of (21) is constant. This condition leads to the ordinary nonlinear differential equation

$$[t_f-f(s)]f''(s)-[f'(s)]^2+1=0,$$

where t_f now is a constant. The general solution of the equation is

$$f(s)=t_f-[(s+c)^2+h^2]^{\frac{1}{2}}, \tag{23}$$

where c, h are arbitrary constants. This is a hyperbola and the parameter h determines its shape. The second parameter c defines the position of the hyperbola apex and can be taken without loss of generality to be zero. Substitution of this function into (22) gives the very important result

$$x(s)=0, \quad y(s)=-h.$$

We can now say that the only contour form which makes the shock wave focus under impact at a single point and at the same instant is the hyperbola (23). In non-dimensional variables h is the distance of the focus from the free surface, t_f is the focusing moment, $h<t_f$. It can be expected that the circular hyperboloid will play the same role in the three-dimensional case.

For some piece of the hyperbola (23) the form of the shock front which is generated by the entry of this piece into the liquid is the circle segment

$$(y+h)^2+x^2=(t_f-t)^2;$$

it can easily be found by substitution of (23) into (22). The pressure jump on the shock front $p_h(\phi,t)$ for the entry of the hyperbola, up to the focusing moment, $0<t<t_f$, is

$$p_h(\phi,t)=\frac{h^{\frac{1}{2}}}{\sin^{\frac{3}{2}}\phi}(t_f-t)^{-\frac{1}{2}}, \tag{24}$$

where $$x=(t_f-t)\cos\phi, \quad y=-h+(t_f-t)\sin\phi.$$

For any geometry of the hyperbola the jump tends to infinity when $\phi\to 0$ and $\phi\to\pi$. But those values can be reached only for the whole hyperbola. In reality there always is some finite piece of it, as shown in the focusing device scheme (see figure 6). In this case

$$\tfrac{1}{2}\pi-\theta<\phi<\tfrac{1}{2}\pi+\theta,$$

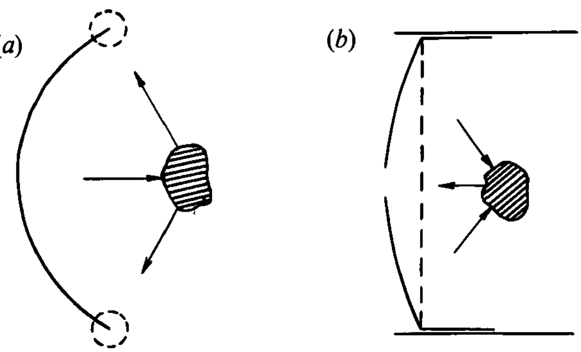

FIGURE 7. Sketch of the loads acting on a stone under the focusing of the shock waves generated (*a*) by the piezoelectrical device and (*b*) by the present one.

where $$\theta = \arcsin[l/l^2+h^2)^{\frac{1}{2}}].$$

The pressure distribution along the shock front is non-uniform, as follows from (24). Its maximum is reached when $\phi = \frac{1}{2}\pi \pm \theta$ and it is given by

$$p_h(\tfrac{1}{2}\pi \pm \theta, t) = \frac{1+\alpha^2}{1-\alpha^2}\left(1-\frac{t}{t_f}\right)^{-\frac{1}{2}}. \qquad (25)$$

Here

$$\alpha = H/l, \quad h = l(1-\alpha^2)/2\alpha, \quad t_f = l(1+\alpha^2)/2\alpha$$

and the contour geometry is determined by the single parameter $\alpha(0 < \alpha < 1)$. It is now clear that to increase the pressure maximum, α must be taken close to unity. Then the hyperbola is close to a wedge with characteristic slope and its focusing distance h is small. This can be considered as a double focusing case. Firstly, the pressure at the contact point increases due to the contact point velocity being close to the sound velocity. Secondly, there is normal focusing at the point of intersection of all rays. To clarify this scheme it is convenient to use the ray method in the spirit of Lesser and start from (20). In a normal focusing apparatus a shock wave is generated by a circular segment which is submerged in a liquid and impacted by a special piezoelectric device. For this approach (20) also can be used with

$$R(s) = -R, \quad f(s) = 0, \quad p_c(s) = 1, \quad t_f = R.$$

In this case (20) has the form

$$p_{sw}(s,t) = (1-t/t_f)^{-\frac{1}{2}}, \qquad (26)$$

where R is the radius of the circle. This focusing approach has been studied in connection with this medical application to kidney stone destruction (see Gronig 1989). In our case $p_c(s)$ is not constant and can be large when α is close to unity. Then, on the right-hand side of (20), not only the ratio of the shock curvature radii at some 'initial' moment and just before the focusing moment (main focusing), but also the 'initial' pressure $p_c(s)$ (additional intensification of the shock), will be high. Comparison of (25), (26) yields that in an energetic sense the present method can be more effective.

A comparison of the loads acting on the stone under the focusing of the shock waves generated by the piezoelectrical device and the present one is shown in figure 7. It is seen that these methods can be considered as opposite to each other. In the first case negative loads are caused by the edge effect but in our case they are caused by the hole at the top of hyperbola, which is necessary for the outflow of air between the liquid surface and the solid contour.

8. Conclusion

In this section some general remarks and open questions are collected, and some further results on the same subject to be published in a near future are also mentioned.

We have presented an analytical description of liquid motion in the narrow zone just behind the shock generated by the entry of a blunt contour into an ideal weakly compressible liquid. In the general case the velocity field and the pressure distribution inside this zone are given by quadratures and depend essentially on the geometry of the entering body. Nevertheless, the characteristics of the flow can be presented in explicit form and can be analysed in detail.

The end of the supersonic stage and the shock wave escape onto the free surface following it is the dramatic period in the water-entry process. The linear theory fails so close to the contact points. It was shown by Korobkin & Pukhnachov (1985) that for an accurate description of the fine flow structures in those regions the nonlinear transonic theory must be used. This nonlinear flow has to be matched with the acoustic flow given by (19) far from the contact points. An analysis of the transonic model and the matching condition will be published. This model is valid for convex entering contours. The situation for complicated bodies can be more difficult. For example, if a body has a point where the contour slope is the characteristic one and the curvature is equal to zero then the asymptotics (19) are not valid and we have to expect higher pressure at this point than that given by the transonic model.

A similar analysis for the three-dimensional case has not been done yet. Nevertheless, the results obtained by Shamgunov (1966) and Lesser (1981) for the axisymmetrical case are a good basis for further investigations.

When the contour slope is small – only this part of the body is responsible for the liquid flow in the region $D(t)$ – capillary effects can be important. It is necessary to estimate their influence on the impact process for optimal construction of a double-focusing device.

The role of viscous forces is not very clear. They are important for understanding liquid motion in the vicinities of the contact points at the supersonic stage and can be responsible for shock escape, and also can be important in the focusing process. Actually, the pressure distribution along the shock front just before the focusing moment is not uniform. This will lead to so great a curvature of the front that it cannot be considered as of Hugoniot type any more (Sichel 1963) and viscous effects will have to be taken into account.

The construction and possible applications of a double-focusing device are not simple. Some effects neglected in the present analysis can be important. Further experimental and theoretical investigations in this field are welcome.

Unfortunately, the physics of kidney stone destruction is not clear yet. It is possible that for optimal destruction some load scheme, which is not as simple as that shown in figure 7, must be used. The present approach could be very helpful in solving the inverse problem, when the load scheme is given and the corresponding geometry responsible for shock initiation has to be found.

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